THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2014-2015
Suggested Solution to Test 2

1. (a) Since $e^z \sin z \cos 2z$ is entire, the integral is zero.

   (b) Let $f(z) = \sin z$, by Cauchy integral formula,

$$\int_C \frac{\sin z}{z}\, dz = \int_C \frac{f(z)}{z}\, dz = 2\pi i f(0) = 0.$$

   (c) Let $f(z) = \dfrac{1}{(z^2-4)e^z}$, by Cauchy integral formula,

$$\int_C \frac{1}{z^2(z^2-4)e^z}\, dz = \int_C \frac{f(z)}{z^2}\, dz = 2\pi i f'(0) = \frac{\pi i}{2}.$$

2. For any $z \in \mathbb{C}$,

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ e^z - 1 &= z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \end{aligned}$$

so, for any $z \neq 0$, we have

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots .$$

Note that the series on the right hand side converges to an analytic function $f(z)$ for all $z \in \mathbb{C}$. By the construction, we know that

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

3. (a) For any $z \in \mathbb{C}$,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

   and for any $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k=0}^{\infty} z^k.$$

   Therefore, for any $|z| < 1$,

$$\begin{aligned} \frac{e^z}{1-z} &= \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{k=0}^{\infty} z^k\right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{r=0}^{k} \frac{1}{r!}\right) z^k \end{aligned}$$

(b) For any $|z| < 1$,

$$\begin{aligned}
\frac{e^z}{1-z} &= \sum_{k=0}^{\infty}\left(\sum_{r=0}^{k}\frac{1}{r!}\right)z^k \\
\frac{d}{dz}\frac{e^z}{1-z} &= \frac{d}{dz}\sum_{k=0}^{\infty}\left(\sum_{r=0}^{k}\frac{1}{r!}\right)z^k \\
\frac{(2-z)e^z}{(1-z)^2} &= \sum_{k=0}^{\infty}\left(\sum_{r=0}^{k}\frac{1}{r!}\right)kz^{k-1}
\end{aligned}$$

4. (a) Let C be the circle $\{|z-z_0| = R\}$ which is positively oriented. Then, by Cauchy integral formula

$$\begin{aligned}
f^{(n)}(z_0) &= \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \\
\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \\
&\leq \frac{n!}{2\pi}\cdot 2\pi R\cdot\frac{M}{R^{n+1}} \\
&= \frac{n!}{R^n}M
\end{aligned}$$

(b) Since f is bounded, there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.
Let $z_0 \in \mathbb{C}$. Since f is an entire function, f is analytic in $\{|z - z_0| \leq R\}$ for all $R > 0$. By using (a), we know that $|f'(z_0)| \leq \frac{M}{R}$. Let R goes to infinity, we can show that $f'(z_0) = 0$. $f'(z) = 0$ for all $z \in \mathbb{C}$ implies that f is a constant function.

(c) Let P be the closed parallelogram spanned by ω_0 and ω_1. The parallelogram P lies inside the interior of the disk $D = \{|z| \leq R\}$. By using the maximum principle, there exists $z_M \in \partial D = \{|z| = R\}$ such that $|f(z_M)| \geq |f(z)|$ for all $z \in D$. In particular, $|f(z_M)| \geq |f(z)|$ for all $z \in P$, i.e. f is bounded in P.
Let $z \in \mathbb{C}$, there exists $z' \in P$ such that $f(z) = f(z')$. Therefore, $|f(z)| = |f(z')| \leq |f(z_M)|$ for all $z \in \mathbb{C}$, i.e. f is bounded on the whole complex plane. By using (b), f is a constant function.